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APPROXIMATIONS FOR THE REPAIRMAN PROBLEM WITH TWO REPAIR FACILITIES. II. SPARES

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13 ABSTRACT

The model considered here consists of n operating units which are subject to stochastic failure according to an exponential failure time distribution. These operating units are backed up by m_n spare units. Failures can be of two types. With probability $p(q)$ a failure is of type 1(2) and is sent to repair facility 1(2) for repair. Repair facility 1(2) operates as a $s_n^1(s_n^2)$-server queue with exponential repair times having parameter $\mu_1(\mu_2)$. The number of units waiting for or undergoing repair each of the two facilities is a continuous-parameter Markov chain with finite state space. This paper derives limit theorems for the stationary distribution of this Markov chain as n becomes large under the assumption that s_n^1, s_n^2, and m_n grow linearly with n. These limit theorems give very useful approximations, in terms of the seven parameters characterizing the model, to a distribution that would be impossible to calculate in practice.

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NONTECHNICAL SUMMARY

This report considers a generalization of the repairman models treated in [3]. Here the system studied consists of n operating units, m_n spare units, and two repair facilities. The operating units are subject to failures of two types: minor and major. Minor failures are sent to a local repair facility and major failures to a central repair facility. Once a unit is repaired it is returned to the spare pool if n units are operating, otherwise it goes directly into operation.

As in [3], we shall assume that all failure and service times have an exponential distribution. The stochastic processes representing the number of units waiting or undergoing repair at each of the two repair facilities is a finite state, continuous-parameter Markov chain, which attains a stationary or steady-state distribution after a long time has elapsed Actual numerical computation of this steady-state distribution is difficult However, the method developed in this paper provides a readily calculated approximation when n is large, along with results which allow for reliable prediction of system performance. Our principal concern in this paper is to understand how system performance is effected by the many system parameters. At the current state of knowledge we feel that this is the correct approach and that attempts to optimize system performance should come latter

For sake of numerical illustration we give several examples Suppose there are 100 operating units and 30 spare units of a single type and that their failure time has exponential distribution with mean 10 days With probability 8 the failure is minor and can be fixed

at the local repair facility, number 1. With probability .2 it is major and must be repaired at the central facility, number 2. Repair facility 1 has 10 servers with mean repair time 1 day. Repair facility 2 has 15 servers with mean repair time 5 days. Let X_i denote the number of units waiting for or undergoing repair at facility i $(i = 1, 2)$ in the steady-state. Then the approximations obtained in this report yield the fact that X_1 and X_2 are independent, X_1 (X_2) is approximately normal with mean 8 (10) and standard deviation 2.8 (3.1). The probability that 100 units are operating is .998. These results are unchanged if the number of spares is reduced to 11. In this example neither of the repair facilities is saturated and essentially a maximal number of units are operating.

In contrast to this first example, suppose the number of servers at facility 1 is reduced from 10 to 5 and that the other parameters remain the same. Then the random 2-vector (X_1, X_2) is approximately a bivariate normal with mean vector (61, 6.2) and covariance matrix $\begin{pmatrix} 69 & -6.2 \\ -6.2 & 6.2 \end{pmatrix}$. The number of operating units is approximately normal with mean 62 and standard deviation 8.7. If in this example, the number of spares is reduced from 30 to 15 units, then the distribution of the number of operating units is uneffected, however (X_1, X_2) is bivariate normal with mean vector (45, 6.2) and the same covariance matrix. Thus in this case adding more spares only increases the congestion at facility one without increasing the number of operating units.

Let n denote the number of operating units, m_n the number of spare units, λ the failure rate, $p(1-p)$ the probability of a minor (major) failure, $s_n^1(s_n^2)$ the number of servers at facility 1(2),

and $\mu_1(\mu_2)$ the service rate of a single server at facility 1(2). The following general conclusions are among those that can be drawn from the results of this paper:

(i) if $\frac{n\lambda p}{\mu_1} < \min(s_n^1, m_n)$

and $\frac{n\lambda q}{\mu_2} < \min(s_n^2, m_n)$,

then both facilities will be unsaturated and essentially a full set of n units will be operating;

(ii) if, on the other hand, either

$$\frac{n\lambda p}{\mu_1} > s_n^1$$

or

$$\frac{n\lambda q}{\mu_2} > s_n^2 ,$$

then at least one facility will be saturated and fewer than n units will be operating.

APPROXIMATIONS FOR THE REPAIRMAN PROBLEM

WITH TWO REPAIR FACILITIES, II: SPARES*

by

Donald L. Iglehart and Austin J Lemoine

1. INTRODUCTION

The first paper in this series [3] considered a generalization of the classical repairman problem in which there were two repair facilities but no provision for spare units. In this paper we study the same model but now include spare units. Again we have n operating units which are subject to failures according to an exponential failure distribution with parameter $\lambda > 0$. Backing up these n operating units are m_n spare units which can be used to replace any of the operating units that fail. At most n units can be operating at a given time. Two types of failures are possible. With probability $p(q)$ a failure of type one (two) occurs and the failed unit requires service from repair facility 1(2) which operates like an s_n^1 (s_n^2) -server queue with exponential service time distribution having parameter μ_1 (μ_2). When repairs are completed on a unit, it returns to the spare pool and is once again available to be used as an operating unit. The flow of units in the system is shown in Figure 1.

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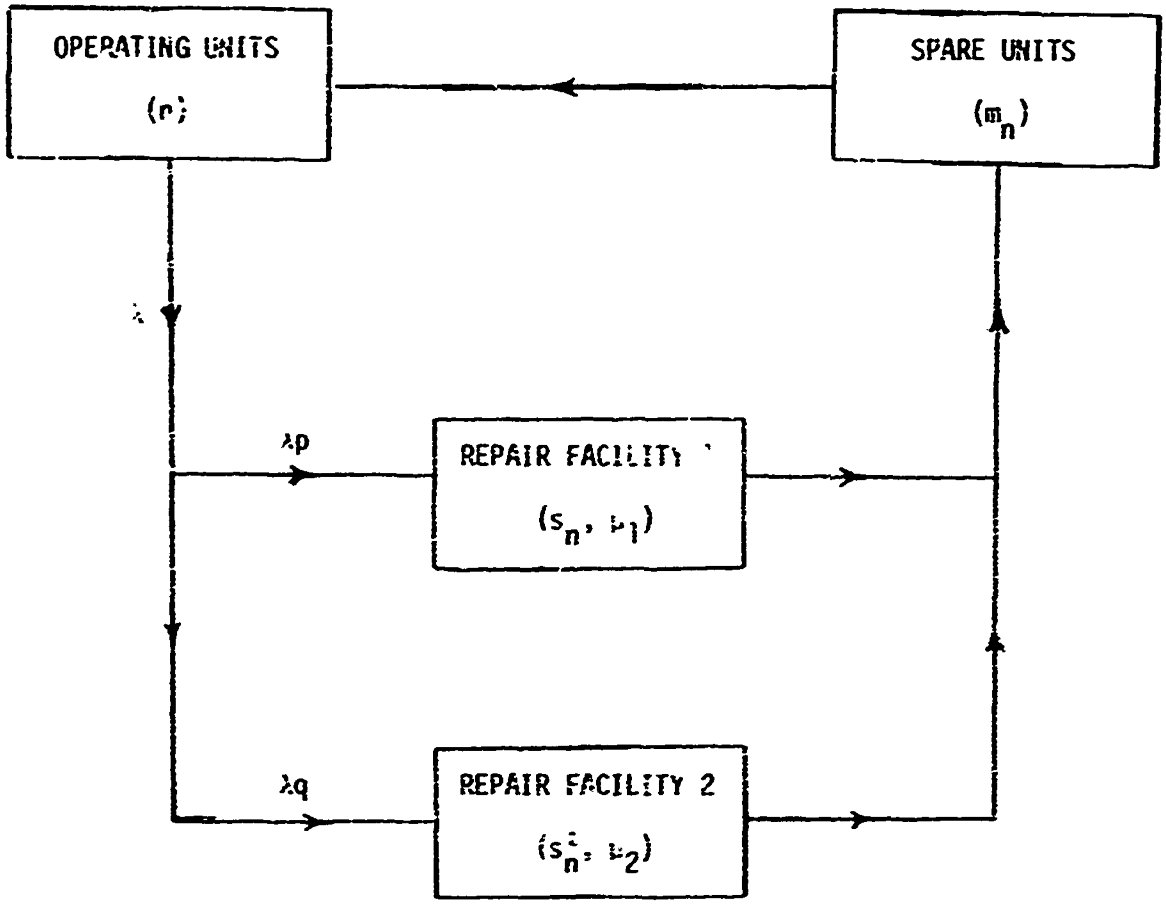

Figure 1

As in [3] we let $X_n^i(t)$ denote the number of units waiting and undergoing repair at facility $i = 1, 2$. The process $\underset{\sim}{X}_n(t) = (X_n^1(t), X_n^2(t))$ is a positive recurrent Markov chain (M.c.) with a finite state space and hence possesses a stationary distribution. Our goal again is to study this stationary distribution as $n \longrightarrow \infty$. This we do under the assumption that $s_n^i \sim ns_i$ $(i = 1, 2)$ and $m_n \sim nm$ as $n \longrightarrow \infty$, where $0 < s_i \leq 1$ and $m > 0$. Our results provide simple approximations for the stationary distribution in terms of the seven independent parameters in the model besides n: λ, p, s_1, μ_1, s_2, μ_2, m. When spares are included in the model, there are a great many cases to consider most of which yield different results. As a complete elaboration of all cases would prove to be

very tedious, we have elected to present only a representative sample of cases since the methodology seems more important than the specific results.

The organization of the paper is as follows. As in [3] we treat the classical repairman problem with one repair facility in Section 2. The full two facility model is dealt with in Section 3.

2. ONE REPAIR FACILITY

First we consider the model in Figure 1 with only one repair facility; i.e., $p = 1$, $q = 0$. Let $X_n(t)$, a birth-death process, denote the number of units waiting or undergoing repair at time t. Let $\mu_1 = \mu$ and $s_n^1 = s_n$ with $s_n \sim ns$, where $0 < s < 1$. The state-space of the process is $N_n = \{0, 1, \ldots, n + m_n\}$, the birth parameters are $\lambda_i^{(n)} = (n - [i-m_n]^+)\lambda$, and the death parameters are $\mu_i^{(n)} = (i \wedge s_n)\mu$. Let X_n denote a random variable (r.v.) having the stationary distribution of the process $\{X_n(t) : t \geq 0\}$ and $Y_n = n - [X_n-m_n]^+$, the stationary number of units operating. If $p_i^{(n)} = P\{X_n = i\}$, then we know from the general theory that

$$p_i^{(n)} = \pi_i^{(n)}/\pi_{N_n}^{(n)} ,$$

where

$$\pi_0^{(n)} = 1, \tag{2.1}$$

$$\pi_i^{(n)} = (\lambda_0^{(n)} \cdots \lambda_{i-1}^{(n)})/(\mu_1^{(n)} \cdots \mu_i^{(n)}), \quad i \geq 1 \tag{2.2}$$

and

$$\pi_A^{(n)} = \sum_{i \in A} \pi_i^{(n)} , \quad A \subset N_n$$

As in [3], we let $\mathcal{P}(\lambda)$ denote a Poisson r.v. with parameter λ, and $\mathcal{B}(n,p)$ denote a binomial r.v. with parameters n and p. Let $a = \lambda/\mu$. Then the first results is

(2.3) PROPOSITION. <u>If</u> $a < s \wedge m$, <u>then</u>

$$\rho_i^{(n)} = [1 - o(n^{-1/2})]\, P\{\mathcal{P}(na) = i\}, \quad 0 \le i \le s_n \wedge m_n$$

and

$$\rho_i^{(n)} \le [1 - o(n^{-1/2})]\, P\{\mathcal{P}(na) = s_n \wedge m_n\}\alpha^{i-s_n \wedge m_n}$$

$$= o(n^{1/2}\alpha^{i-s_n \wedge m_n}), \quad s_n \quad m_n < i \le n + m_n,$$

<u>where</u> $\alpha = a[(s \wedge m) - \varepsilon] < 1$ <u>(with</u> $\varepsilon > 0$ <u>chosen suitably small) and the terms</u> $o(n^{-1/2})$ <u>are uniform in</u> $i \in N_n$.

<u>Proof.</u> We shall only treat the case $s \le m$, the other being similar. Then from (2.1) and (2.2) we can write

$$\pi_i^{(n)} = e^{na}\, P\{\mathcal{P}(na) = i\}, \qquad 0 \le i \le s_n,$$

$$\pi_i^{(n)} = e^{na}\, P\{\mathcal{P}(na) = s_n\}\, \left(\frac{na}{s_n}\right)^{i-s_n}, \quad s_n < i \le m_n,$$

and

$$\pi_i^{(n)} = e^{na}\, P\{\mathcal{P}(na) = s_n\}\, \left(\frac{na}{s_n}\right)^{m_n - s_n} \prod_{k=0}^{i-m_n-1} \left(\frac{n-k}{s_n} \cdot a\right), \quad m_n < i \le n + m_n.$$

Using the same method employed in Proposition 2.21 of [3], we can show that

$$e^{-na}\, {}^{(n)}_{N_n} = 1 + o(n^{-1/2}), \quad \text{as } n \longrightarrow \infty.$$

From this (2.3) follows immediately.

From (2.3) we obtain the following corollary using the classical limit laws for $\mathcal{P}(na)$.

(2.4) COROLLARY. *If* $a < s \wedge m$, *then*

$$\lim_{n\to\infty} P\{X_n \le s_n \wedge m_n\} = 1,$$

$$n^{-1} X_n \Longrightarrow a,$$

$$(X_n - na)/(na)^{1/2} \Longrightarrow N(0,1),$$

$$\lim_{n\to\infty} P\{Y_n = n\} = 1,$$

$$(na)^{1/2} P\{X_n = k\} - (2\pi)^{-1/2} \exp\{-z_{nk}/2\} \longrightarrow 0,$$

uniformly for integers $k \in R_+$, *where* $z_{nk} = (k-na)/(na)^{1/2}$.

(2.5) REMARK. *This result shows that when* $a < s \wedge m$ *and* n *is large the number of busy servers fluctuates about* na *and with very high probability* n *units are operating and no queue forms at the repair facility. One would normally expect* $s < m$; *however, just so long as* $m > a$ *the exact number of spares is not important. The condition* $a/s < 1$ *is a light traffic condition which seems to be a desirable state of affairs from an operational point-of-view.*

Next we consider two cases of heavy traffic $(a > s)$ before returning to the remaining case of light traffic $(m < a < s)$, which requires a different technique.

(2.6) PROPOSITION. If $a > s/[1 - (s-m)^+]$, then

$$\rho_i^{(n)} = [1 - o(n^{-1/2})]\, P\{\mathcal{P}(s_n b) = n + m_n - i\}, \quad m_n \vee s_n \le i \le n + m_n,$$

and

$$\rho_i^{(n)} = o(n^{-1/2}\, \beta^{m_n \vee s_n - i}) \qquad 0 \le i < m_n \vee s_n$$

where $b = a^{-1}$ and $\beta = (\frac{s+\varepsilon}{a})\left[\frac{1}{1-(s-m)^+}\right] < 1$ (with $\varepsilon > 0$ chosen suitably small) and the terms $o(n^{-1/2})$ are uniform in $i \in N_n$.

Proof. We only treat the case $s \le m$, the other case being similar again. For this case it is more convenient to start with $\pi_{n+m_n}^{(n)} = 1$; see the proof of Proposition (2.21) in [3] for further discussion. Using this normalization we can show that

$$e^{-s_n b}\, \pi_{N_n}^{(n)} = 1 + o(n^{-1/2}), \quad \text{as} \quad n \longrightarrow \infty.$$

Once this is established the proof follows the general method used in [3].

As we have seen above the following corollary is obtained immediately.

(2.7) COROLLARY. If $a > s/[1-(s-m)^+]$, then

$$\lim_{n\to\infty} P\{X_n \ge m_n \vee s_n\} = 1 \qquad \text{and}$$

$$n^{-1} X_n \Longrightarrow 1 + m - sb.$$

In addition, if $|n^{-1}s_n - s| = o(n^{-1/2})$ and $|n^{-1}m_n - m| = o(n^{-1/2})$, then

$$[X_n - n(1 + m - sb)]/(nsb)^{1/2} \Longrightarrow N(0,1),$$

$$(Y_n - nsb)/(nsb)^{1/2} \Longrightarrow N(0,1), \quad \underline{\text{and}}$$

$$(nsb)^{1/2} P\{Y_n = k\} - (2\pi)^{-1/2} \exp\{-z_{nk}/2\} \longrightarrow 0,$$

uniformly for integers $k \in R_+$, where $z_{nk} = (k-nsb)/(nsb)^{1/2}$.

(2.8) REMARK. Notice that in this case roughly nsb units will be operating regardless of the level of spares. Furthermore, all servers will be occupied and queues of the order of $n[1 + m - s(b+1)]$ will form. The moral of the story here is that if the queue is in a heavy traffic condition $(a > s/[1 - (s-m)^+])$ spares are of no help in eliminating congestion and down operating units.

Finally, we turn to the last case of light traffic, $m < a < s$. Let

(2.9) PROPOSITION. If $m < a < s$ and $[e(a+ma)/(m+ma)]^m (1+a)^{-1} = \gamma < 1$, then

$$\rho_i^{(n)} = [1 - o(n^{-1/2})] P\{\mathcal{B}(n+m_n, a/(1+a)) = i\}, \quad m_n \le i \le s_n,$$

$$\rho_i^{(n)} \le [1 - o(n^{-1/2})][(\tfrac{1+m}{1+a})^m (\tfrac{e^a}{1+a})]^n P\{\mathcal{P}(na) = i\}, \quad 0 \le i < m_n,$$

and

$$\rho_i^{(n)} = o[n^{-1/2}\left(\frac{1+m-s+\varepsilon}{s} a\right)^{i-s_n}], \quad s_n < i \le n + m_n$$

where $\varepsilon > 0$ is chosen suitably small and the term $o(n^{-1/2})$ are uniform in $i \in N_n$.

Proof. Here we take the $\pi_i^{(n)}$'s to be normalized so that $\pi_0^{(n)} = 1$. Then we find that

$$\pi_i^{(n)} = e^{na} P\{\mathcal{P}(na) = i\} \qquad , 0 \le i < m_n,$$

$$\pi_i^{(n)} = b_n P\{\mathcal{B}(n+m_n, a(1+a)^{-1}) = i\} \qquad , m_n \le i \le s_n,$$

and

$$\pi_i^{(n)} = b_n P\{\mathcal{B}(n+m_n, a(1+a)^{-1}) = s_n\} c_i^{(n)}, \; s_n < i \le n + m_n$$

where

$$b_n = \frac{n^{m_n} (1+a)^{n+m_n}}{(n+m_n) \cdots (n+1)},$$

and

$$c_i^{(n)} = \left(\frac{n + m_n - s_n}{s} \cdot a\right) \cdots \left(\frac{n + m_n - i + 1}{s_n} \cdot a\right), \; s_n < i \le n + m_n.$$

First we shall show that

$$(2.10) \qquad b_n^{-1} \pi_{N_n}^{(n)} = 1 + o(n^{-1/2}), \quad \text{as} \quad n \longrightarrow \infty$$

Using Chebyshev's inequality it is easy to show that

$$(2.11) \qquad b_n^{-1} \sum_{i=m_n}^{s_n} \pi_i^{(n)} \ge 1 - O(n^{-1}).$$

Also using the fact that $(1+m-s)a < s$ by hypothesis, we can show that

$$(2.12) \qquad b_n^{-1} \sum_{i=s_n+1}^{n+m_n} \pi_i^{(n)} = o(n^{-1/2}).$$

For n sufficiently large it follows from results on exponential convergence that

$$P\{\mathcal{P}(na) < m_n\} \le [e^{m-a}\,(a/m)^m]^n \quad ;$$

see for example HEATHCOTE (1967), p. 224. Hence

$$b_n^{-1} \sum_{i=0}^{m_n-1} \pi_i^{(n)} \le e^{na}\, b_n^{-1}\, [e^{m-a}(a/m)^m]^n \; .$$

It is easy to show for large n that

$$e^{na}\, b_n^{-1} < \left[\left(\frac{1+m}{1+a}\right)^m \left(\frac{e^a}{1+a}\right)\right]^n$$

Thus

$$(2.13) \qquad b_n^{-1} \sum_{i=0}^{m_n-1} \pi_i^{(n)} < \gamma^n = o(n^{-1/2}).$$

Combining (2.11), (2.12), and (2.13) yields (2.10). The rest of the proof then follows immediately.

Again the next corollary follows immediately

(2.14) COROLLARY. <u>If</u> $m < a < s$ <u>and</u> $\gamma < 1$, <u>then</u>

$$\lim_{n\to\infty} P\{m_n \le X_n \le s_n\} = 1$$

$$n^{-1}\, X_n \Longrightarrow a(1+m)/(1+a), \text{ and}$$

$$n^{-1}\, Y_n \Longrightarrow (1+m)/(1+a).$$

<u>In addition, if</u> $|n^{-1}\, m_n - m| = o(n^{-1/2})$, <u>then</u>

$$[X_n - na(1+m/(1+a)]/[n(1+m)a(1+a)^{-2}]^{1/2} \Longrightarrow N(0,1)$$

<u>and</u> .

$$[Y_n - n(1+m)/(1+a)]/[n(1+m)a(1+a)^{-2}]^{1/2} \Longrightarrow N(0,1).$$

(2.15) REMARK. The condition $\gamma < 1$ requires that $m(<a)$ not be "too close" to a. For this case there will be with high probability idle capacity at the repair facility but less than n units operational. In this case, however, it would pay to add more spares. Fortunately, in practice it seems unlikely that there would be more repairman that spares, expect for those instances where the individual units are very expensive.

This completes our discussion of the one repair facility model and we now move to the two facility model.

3. TWO REPAIR FACILITIES

In this section we shall treat the two facility model illustrated in Figure 1. As was indicated in the introduction, we shall only consider a sample of the cases which seem to be of greatest practical interest. Even for these cases we only sketch the proof. Recall that $\underset{\sim}{X}_n(t) = (X_n^1(t), X_n^2(t))$, where $X_n^i(t)$ represents the number of units waiting and undergoing repair at facility i, is a positive recurrent M.c. with finite state space $\Delta_n = \{(i,j) : i, j \geq 0, \ i + j \leq n + m_n\}$. As was the case in the first paper [3], $\{\underset{\sim}{X}_n(t) : t \geq 0\}$ is a reversible competition process (r.c.p.); see [2, p. 331] for details. This fact again allows us to define constants $\{\pi_{ij}^{(n)} : (i,j) \in \Delta_n\}$ such that the $\lim_{t\to\infty} P\{\underset{\sim}{X}_n(t) = (i,j)\} = P\{\underset{\sim}{X}_n = (i,j)\} = \rho_{ij}^{(n)}$, where

$$\rho_{ij}^{(n)} = \pi_{ij}^{(n)}/\pi_{\Delta_n}^{(n)} \qquad (i,j) \in \Delta_n \tag{3.1}$$

and

$$\pi_A^{(n)} = \sum_{(i,j)\in A} \pi_{ij}^{(n)} \quad \text{for } A \subset \Delta_n.$$

Again we let $Y_n = n - [X_n^1 + X_n^2 - m_n]^+$ denote the stationary number of units operating and $N(\underset{\sim}{0}, \underset{\sim}{\Sigma})$ stand for a normal random vector with mean vector $\underset{\sim}{0}$ and covariance matrix $\underset{\sim}{\Sigma}$. Throughout the rest of this paper we assume $s_n \sim ns_i$ $(i = 1, 2)$ and $m_n \sim nm$ as $n \longrightarrow \infty$, where $0 < s_i < 1$ and $m > 0$. Let $a_1 = \lambda p/\mu_1$, $a_2 = \lambda q/\mu_2$, and $\underset{\sim}{a} = (a_1, a_2)$. The first case we treat is perhaps the most favorable one from the point-of-view of an operational system. Both facilities are operating in light traffic and an ample number of spares are available.

(3.2) PROPOSITION. <u>If</u> $s_1 + s_2 \leq m$, $a_1 < s_1$, and $a_2 < s_2$ <u>then</u>

$$\rho_{ij}^{(n)} = [1 - o(n^{-1/2})] \, P\{\mathcal{P}(na_1) = i\} \cdot P\{\mathcal{P}(na_2) = j\},$$

<u>where</u> $0 \leq i \leq s_n$ <u>and</u> $0 \leq j \leq s_n$.

<u>Proof.</u> Set $\pi_{00}^{(n)} = 1$ and use the method of [2] to obtain

$$\pi_{ij}^{(n)} = e^{n(a_1+a_2)} \, P\{\mathcal{P}(na_1) = i\} \cdot P\{\mathcal{P}(na_2) = j\} \tag{3.3}$$

for $0 \leq i \leq s_n^1$ and $0 \leq j \leq s_n^2$. Using the appropriate expressions for the other $\pi_{ij}^{(n)}$'s we can show that $e^{-n(a_1+a_2)}\pi_{\Delta_n}^{(n)} = 1 + o(n^{-1/2})$ as $n \longrightarrow \infty$. Combining this result with (2.18) yields (2.17).

Again the proof of the following corollary follows immediately.

(3.4) COROLLARY. <u>If</u> $s_1 + s_2 \leq m$, $a_1 < s_1$, <u>and</u> $a_2 < s_2$, <u>then</u>

$$\lim_{n\to\infty} P\{\underset{\sim}{X}_n \leq (s_n^1, s_n^2)\} = 1; \tag{3.5}$$

$$n^{-1} \underset{\sim}{X}_n \Longrightarrow \underset{\sim}{a}; \tag{3.6}$$

$$(3.7) \qquad (\underset{\sim}{X}_n - n\underset{\sim}{a})/n^{1/2} \Longrightarrow N(\underset{\sim}{0}, \underset{\sim}{\Sigma}),$$

where

$$\underset{\sim}{\Sigma} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix};$$

and

$$(3.8) \qquad \lim_{n\to\infty} P\{Y_n = n\} = 1.$$

(3.9) REMARK. Under the conditions of (3.4) no queues form and n units are operating with high probability. Note also that X_n^1 and X_n^2 are asymptotically independent. The two facility model in this case behaves exactly like two independent one facility models.

Before proceeding to the next result, we mention the fact that the conclusions of (3.2) and (3.5) are essentially unchanged if one of the following three situations hold:

$$(3.10) \qquad s_1 + s_2 > m, \ a_1 < s_1 \le m, \ a_2 < s_2 \le m \text{ and either } m > s_1 + a_2 \text{ or } m > s_2 + a_1;$$

$$(3.11) \qquad a_1 < s_1 \le m, \ a_2 < m \le s_2; \text{ or}$$

$$(3.12) \qquad a_1 < m < s_1, \ a_2 < m < s_2.$$

In all of these cases (3.6), (3.7), and (3.8) hold, however, (3.5) requires slight modification. Observe that in all of these cases we need $a_i < s_i \wedge m$, $i = 1, 2$. It is this condition which insures a light traffic situation at both facilities and n units operating with high probability.

Now let $\nu = (1+a_2)/a_1$ and $\gamma = a_2/(1+a_2)$. The next case we consider is one in which facility 1(2) is in heavy (light) traffic and an ample number of spares are available.

(3.13) PROPOSITION. *If* $s_1 + s_2 \leq m$, $a_1 > s_1\,(1+a_2)$, *and* $a_2 < s_2$, *then*

$$\rho_{ij}^{(n)} = [1 - o(n^{-1/2})]\; P\{\mathcal{P}(s_n^1\nu) = n + m_n - i\}\; P\{\mathcal{B}(n + m_n - i, \gamma) = j\},$$

where $m_n \leq i$, $0 \leq j \leq s_n$, *and* $i + j \leq n + m_n$.

Proof. Set $\pi_{n+m_n,0}^{(n)} = 1$. Then again using the method of [3] we obtain

$$\pi_i^{(n)} = e^{s_n^1\nu}\; P\{\mathcal{P}(s_n^1\nu) = n + m_n - i\}\; P\{\mathcal{B}(n + m - i) = j\},$$

for $m_n \leq i$, $0 \leq j \leq s_n^2$, and $i + j \leq n + m_n$. The rest follows using the standard approach.

In this case we have

(3.14) COROLLARY. *If* $s_1 + s_2 \leq m$, $a_1 > s_1\,(1+a_2)$, *and* $a_2 < s_2$, *then*

(3.15) $$\lim_{n\to\infty} P\{X_n^1 \geq m_n,\ X_n^2 \leq s_n^2\} = 1;$$

(3.16) $$n^{-1}\,\underset{\sim}{X}_n \Longrightarrow \underset{\sim}{r}_1 = (1 + m - s_1\nu,\ \gamma\nu s_1);$$

(3.17) $$n^{-1}\,Y_n \Longrightarrow s_1\nu(1-\gamma) < (1+a_2)^{-1}.$$

In addition, if $|n^{-1}\,s_n - s_1| = o(n^{-1/2})$ *and* $|n^{-1}\,m_n - m| = o(n^{-1/2})$, *then*

(3.18) $$(\underset{\sim}{X}_n - n\underset{\sim}{r}_1)/n^{1/2} \Longrightarrow N(\underset{\sim}{0},\underset{\sim}{\Gamma}),$$

where

$$\underset{\sim}{\Gamma} = \begin{pmatrix} s_1\nu & -s_1\nu\gamma \\ -s_1\nu\gamma & s_1\nu\gamma \end{pmatrix}; \quad \textit{and}$$

$$(3.19) \qquad [Y_n - ns_1\nu(1-\gamma)]/[ns_1\nu(1+\gamma)]^{1/2} \Longrightarrow N(0,1).$$

(3.20) REMARK. *In this case with high probability repair facility* 1 *is completely occupied and no queue forms at repair facility* 2. *Less than* n *units are operating with high probability regardless of how many spares are provided just so long as* $s_1 + s_2 \leq m$. *Again we see that spares are of no help in alleviating a heavy traffic condition at one of the repair facilities. Adding more spares only increases the congestion at facility* 1 *without producing more operating units.*

The results of (3.13) and (3.14) are essentially unchanged if one of the following three conditions holds:

$$(3.21) \qquad s_1 + s_2 > m, \quad s_1 \leq m, \quad s_2 \leq m$$
$$(1+a_2)s_1 < a_1, \quad a_2 < s_2;$$

$$(3.22) \qquad s_1 \leq m, \quad s_2 > m,$$
$$(1+a_2)s_1 < a_1, \quad a_2 < m; \quad \text{or}$$

$$(3.32) \qquad s_1 > m, \quad s_2 > m$$
$$[(1+a_2)m] \vee s_1 < a_1, \quad a_2 < m.$$

In all of these cases (3.16), (3.17), and (3.18) and (3.14) hold, however, (3.15) requires slight modification. For these cases facility 1 (2) is in heavy (light) traffic. Increasing the level of spares only adds to the congestion at facility 1 without producing more operating units.

The final case we treat is comparable to (2.9). There is idle capacity at both facilities, however, less than n units are operating. As in [3] we let $\mathcal{J}$ denote a trinomial r.v., $p_i = a_i/(1+a_1+a_2)$, $i = 1, 2$, and $\underset{\sim}{p} = (p_1, p_2)$.

(3.24) PROPOSITION. <u>If</u> $(1+a_2)m < a_1 < s_1$, $a_2 < m < s_2$,

<u>and</u>

$$[e(a_1 + ma_1)/(m + ma_1)]^m(1+a_1+a_2)^{-1} = \gamma_1 < 1,$$

<u>then</u>

$$\rho_i^{(n)} = [1 - o(n^{-1/2})] P\{\mathcal{J}(n+m_n; p_1, p_2) = (i,j)\},$$

<u>where</u> $m_n \leq i \leq s_n$, $0 \leq j \leq m_n$.

<u>Proof</u>. Set $\pi_{00}^{(n)} = 1$ and then for $m_n \leq i \leq s_n$, $0 \leq j \leq m_n$ we have

$$\pi_{ij}^{(n)} = d_n P\{\mathcal{J}(n+m_n; p_1,p_2) = (i,j)\},$$

where

$$d_n = \frac{n^{m_n} (1+a_1+a_2)^{n+m_n}}{(n+m_n) \cdots (n+1)} .$$

From here the method is the same.

(3.25) COROLLARY. <u>If</u> $(1+a_2)m < a_1 < s_1$, $a_2 < m < s_2$, <u>and</u> $\gamma_1 < 1$, <u>then</u>

$$\lim_{n\to\infty} P\{m_n \le X_n^1 \le s_n,\ \ 0 \le X_n^2 \le m_n\} = 1,$$

$$n^{-1} \underset{\sim}{X}_n \Rightarrow (1 + m)\underset{\sim}{p}, \qquad \underline{\text{and}}$$

$$n^{-1} Y_n \Rightarrow (1 + m)/(1 + a_1 + a_2)$$

<u>In addition, if</u> $|n^{-1} m_n - m| = o(n^{-1/2})$, <u>then</u>

$$(\underset{\sim}{X}_n - n(1 + m)\underset{\sim}{p})/n^{1/2} \Rightarrow N(\underset{\sim}{0}, \Lambda),$$

<u>where</u>

$$\Lambda = \begin{pmatrix} p_1(1 - p_1) & -p_1p_2 \\ -p_1p_2 & p_2(1 - p_2) \end{pmatrix},$$

and

$$[Y_n - n(1 + m)/(1 + a_1 + a_2)]/[p_1(1 - p_1) + p_2(1 - p_2)]^{1/2} \Rightarrow N(0,1).$$

(3.26) REMARK. <u>In this situation, both facilities are in light traffic but fewer than n units are operating. This again is the situation in which it would pay to buy more spares.</u>

This concludes our discussion of the two facility model. We repeat the principal lessons learned. <u>In order to insure unsaturated conditions at both facilities and a full set of n units operating we need to have</u> $a_1 < s_1 \wedge m$ and $a_2 < s_2 \wedge m$. <u>If on the other hand either</u> $a_1 > s_1$ or $a_2 > s_2$, <u>then at least one facility will be</u>

saturated and fewer than n units will be operating. In this latter case addition of spare units only increases confestion without contributing to operating units.

REFERENCES

[1] HEATHCOTE, C. (1967). Complete exponential convergence. J. Appl. Probability. 4 217-256.

[2] IGLEHART, D.L. (1964). Reversible competition processes. Z. Wahrscheinlichkeitstheorie 2 314-331.

[3] IGLEHART, D.L. and LEMOINE A. (1972). Approximations for the repairman problem with two repair facilities, I: no spares. Technical Report No. 266-2, Control Analysis Corporation, Palo Alto, California.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Birth and death processes						
Central limit theorem						
Limit theorems						
Local central limit theorems						
Logistics systems						
Repairman systems						
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